

Hopf maps as static solutions of the complex eikonal equation

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Abstract

We demonstrate that a class of torus-shaped Hopf maps with arbitrary linking number obeys the static complex eikonal equation. Further, we explore the geometric structure behind these solutions, explaining thereby the reason for their existence. As this equation shows up as an integrability condition in certain non-linear field theories, the existence of such solutions is of some interest.

1 Introduction

In this article we want to report on a class of Hopf maps with arbitrary linking number, which are, at the same time, static solutions to the complex eikonal equation. Further, we want to explore the geometric structure which is behind these solutions and explains, in fact, their existence.

The eikonal equation reads

$$(\partial^\mu \chi)(\partial_\mu \chi) = 0 \tag{1}$$

and describes, for a real scalar function χ , the propagation of wave fronts (field discontinuities) in Minkowski space. Its generalization to complex χ has some applications in optics and quantum mechanics, as well as in general relativity (see [1] and the literature cited there). In [1] an algebraic procedure (based on twistor methods) for the construction of complex solutions to Eq. (1) was developed, and some examples of singular solutions were provided. The complex eikonal equation admits even static solutions, i.e., solutions to the equation

$$(\nabla \chi) \cdot (\nabla \chi) = 0, \tag{2}$$

in contrast to the case of real χ .

The complex eikonal equation (1) has also appeared, in quite a different context, as an integrability condition in some non-linear field theories. In the last few years there has been rising interest in integrable field theories in higher (i.e., more than two) dimensions, where solitonic solutions are, in many cases, provided by certain Hopf maps, see e.g. [2, 3, 4, 5]. In addition, some non-linear field theories which are, in general, not integrable, contain *integrable subsectors* where certain integrability conditions are satisfied. Specifically, the complex eikonal equation (1) defines integrable subsectors in the Skyrme and Skyrme–Faddeev models ([6, 7]). For static, solitonic solutions, this condition reduces to the static complex eikonal equation (2). Finite energy solitons in these integrable subsectors correspond to static solutions defined on one-point compactified \mathbb{R}^3 and may, therefore, be identified with functions on S^3 via stereographic projection. Further, the target space of the fields χ in the integrable subsectors can be identified with the Riemann sphere S^2 (i.e., χ is a holomorphic variable on \mathbb{C}). Therefore, the fields χ in the integrable subsectors of these models are Hopf maps $S^3 \rightarrow S^2$, and can be classified by the Hopf index (the homotopy group $\pi_3(S^2) = \mathbb{Z}$). Consequently, solutions of the complex, static eikonal equation which are, at the

same time, Hopf maps, are of some interest for these non-linear field theories, because they provide finite energy field configurations in their integrable subsectors.

In addition, the static complex eikonal equation (2) has appeared as an integrability condition for the existence of multiple zero modes of the static, Abelian Dirac operator ([8, 9, 10]). In this case, solutions χ to Eq. (2) are again required to be Hopf maps. The Hopf maps described below will indeed give rise to the construction of new classes of zero modes with new and interesting properties, but this issue shall be discussed elsewhere.

In Section 2 we show that certain toroidal Hopf maps $\chi^{(m,n)}$ obey the static eikonal equation for arbitrary integer m and n (here m and n count the number of times the level curves of χ wrap around the two circular directions of a certain torus). Further, we briefly discuss the symmetries of the static eikonal equation, which enables us to construct new solutions from the ones just mentioned.

In Section 3 we explain the geometric structure which lies behind the existence of these solutions. It turns out that the solutions of Section 2 may be understood as pullbacks of trivial solutions of the complex eikonal equation in two dimensions which preserve some metric properties, providing thereby non-trivial three-dimensional solutions. Further, we give a sufficient condition for the existence of solutions of the geometric type discussed in this paper.

2 The solutions

In the sequel, we will express Hopf maps as complex-valued functions which depend on three variables like, e.g., (x, y, z) . Here, the space spanned by these variables may be interpreted either as one-point compactified \mathbb{R}^3 or as the three-sphere S^3 , where a stereographic projection has been performed. The solutions to the static eikonal equation described below do not depend on this interpretation, i.e., they may be interpreted as solutions on \mathbb{R}^3 or on S^3 . This result is related to the fact that the metrics on \mathbb{R}^3 and S^3 are conformally equivalent (i.e., equal up to a local, space-dependent scale transformation), as will become clear in the next section.

The simplest Hopf map is

$$\chi^{(1,1)} = i \frac{2(x + iy)}{2z + i(r^2 - 1)} \quad (3)$$

(the meaning of the superscript $(1, 1)$ is explained below in Eq. (10)). Further, $r^2 \equiv x^2 + y^2 + z^2$, and the irrelevant pre-factor i has been chosen for later convenience. The simplest Hopf map is well-known to obey the static eikonal equation (2), see e.g. [8]. Before demonstrating this fact, we want to introduce toroidal coordinates (η, ξ, φ) via

$$\begin{aligned} x &= q^{-1} \sinh \eta \cos \varphi, & y &= q^{-1} \sinh \eta \sin \varphi \\ z &= q^{-1} \sin \xi; & q &= \cosh \eta - \cos \xi. \end{aligned} \quad (4)$$

Further, we need the gradient in terms of the toroidal coordinates,

$$\nabla = (\nabla \eta) \partial_\eta + (\nabla \xi) \partial_\xi + (\nabla \varphi) \partial_\varphi = q(\hat{e}_\eta \partial_\eta + \hat{e}_\xi \partial_\xi + \frac{1}{\sinh \eta} \hat{e}_\varphi \partial_\varphi) \quad (5)$$

where $(\hat{e}_\eta, \hat{e}_\xi, \hat{e}_\varphi)$ form an orthonormal frame in \mathbb{R}^3 . In terms of toroidal coordinates, the simplest Hopf map reads

$$\chi^{(1,1)} = \sinh \eta e^{i\varphi + i\xi}. \quad (6)$$

Here, surfaces of $\eta = \text{const.}$ are tori in \mathbb{R}^3 . These tori are rotation symmetric around the z axis, and all of them enclose the circle $C = \{\vec{x} \in \mathbb{R}^3 : z = 0 \wedge r^2 = 1\}$. The coordinates φ and ξ are angular coordinates along the two circular directions on each torus. Each level curve of $\chi^{(1,1)}$ (i.e., each curve $\chi^{(1,1)} = \text{const.}$) is located on one torus. It is, in fact, a circle that winds once around each circular direction of the torus. Further, any two different level curves are linked with linking number one, and this linking number is the geometric definition of the Hopf index (which is equal to one for the simplest Hopf map (3)).

For a simple demonstration of the fact that the Hopf map (6) really obeys the eikonal equation it is useful to re-express a general Hopf map χ in terms of two real functions (modulus S and phase σ) like

$$\chi = S e^{i\sigma}. \quad (7)$$

In terms of these real functions, the static eikonal equation (2) leads to the conditions

$$(\nabla S) \cdot (\nabla \sigma) = 0, \quad (\nabla S)^2 = S^2 (\nabla \sigma)^2. \quad (8)$$

For the simplest Hopf map (6) we find, with $S = \sinh \eta$, $\sigma = \xi + \varphi$,

$$\nabla S = q \cosh \eta \hat{e}_\eta, \quad \nabla \sigma = q \left(\hat{e}_\xi + \frac{1}{\sinh \eta} \hat{e}_\varphi \right) \quad (9)$$

which indeed obey Eqs. (8). The important point here is that the equations (8) are expressed only in terms of the target space coordinates S and σ , making the problem essentially two-dimensional. This is precisely what happens for the simplest Hopf map. The factor q , which is present in (9) and cannot be expressed in terms of the target space coordinates, cancels in the relations (8).

A simple generalization to higher Hopf maps is provided by the functions

$$\chi^{(m,n)} = f(\eta) e^{in\varphi + im\xi}, \quad m, n \in \mathbb{Z} \quad (10)$$

which are true Hopf maps if the real function f obeys certain regularity conditions like, e.g., $f(0) = 0$ and $f(\infty) = \infty$ (what we assume in the sequel). The level curves of these Hopf maps still lie on the same tori as above, but now they wind n times around the φ direction and m times around the ξ direction. Further, the Hopf index N_H (i.e., the linking number of any two different level curves) is $N_H = nm$.

We find for the gradient

$$\nabla \chi^{(m,n)} = q e^{im\xi + in\varphi} \left(f' \hat{e}_\eta + im f \hat{e}_\xi + \frac{in}{\sinh \eta} f \hat{e}_\varphi \right), \quad (11)$$

where $f' \equiv \partial_\eta f$, and Eq. (2) leads to the simple differential equation

$$\frac{f'}{f} = \left(m^2 + \frac{n^2}{\sinh^2 \eta} \right)^{\frac{1}{2}} \quad (12)$$

with the solution

$$f = \sinh^{|n|} \eta \frac{\left(|m| \cosh \eta + \sqrt{n^2 + m^2 \sinh^2 \eta} \right)^{|m|}}{\left(|n| \cosh \eta + \sqrt{n^2 + m^2 \sinh^2 \eta} \right)^{|n|}}. \quad (13)$$

These solutions are genuine Hopf maps for all non-zero, integer m, n , because f obeys $f(0) = 0$, $f(\infty) = \infty$.

At this point it is of interest to briefly consider the symmetries of the complex static eikonal equation (2). This will lead to some further understanding of these solutions and allow to construct more solutions from the ones obtained so far. The symmetry group of equation (2) is a direct product of base space and target space symmetries, where the group of base space symmetries is the conformal group in three-dimensional Euclidean space. The group of target space symmetries is given locally by the maps $\chi \rightarrow F(\chi)$, where F is an arbitrary complex function of χ , but not of its complex conjugate $\bar{\chi}$. The requirement that the solutions $\chi' = F(\chi)$ are single-valued again restricts the allowed functions $F(\cdot)$ to the set of holomorphic functions on \mathbb{C} .

The presence of the conformal symmetry on base space implies that the ansatz (10) is an “educated guess” for a solution to Eq. (2) in the sense of the Lie theory of symmetry. That is to say, if we choose a rotation about the z axis and a certain combination of proper conformal transformation along the z axis and translation along the z axis as a maximal set of two commuting base space transformations, then the corresponding infinitesimal symmetry generators (vectors \mathbf{v}^i) are precisely given by the tangent vectors along φ and ξ , $\mathbf{v}^1 = \partial_\varphi$, and $\mathbf{v}^2 = \partial_\xi$. The ansatz (10) is invariant under a combination of these base space transformations and phase transformations of the target space variable χ , i.e., under the action of the vector fields $\tilde{\mathbf{v}}^1 = \partial_\varphi - im\chi\partial_\chi$ and $\tilde{\mathbf{v}}^2 = \partial_\xi - im\chi\partial_\chi$, which provides precisely the educated guess according to Lie. A concise discussion of these points can be found in Ref. 4, where the symmetries of an integrable model with infinitely many Hopf solitons are discussed in detail.

Further, we may use the target space symmetries to construct more solutions from the ones given in (13). In fact, each field $\chi' = F(\chi)$ is a solution, where χ is a solution and F is a holomorphic function on \mathbb{C} .

3 Geometric background

Here we want to explain the geometric structure behind the solutions (13), which will, in fact, allow to understand the reason why they exist. For this purpose, let us first observe that there exist trivial solutions to the complex eikonal equation in \mathbb{R}^2 or, equivalently, in \mathbb{C} . Indeed, for real, cartesian

coordinates $(u, v) \in \mathbb{R}^2$ with $w = u + iv$ and gradient

$$\nabla^{(2)} \equiv \hat{e}_u \partial_u + \hat{e}_v \partial_v \quad (14)$$

the complex eikonal equation $(\nabla^{(2)} f(w))^2 = 0$ is equivalent to the Cauchy-Riemann equations, which are obeyed by arbitrary holomorphic functions $f(w)$. So, obviously, the complex coordinate $w = u + iv$ itself obeys the eikonal equation,

$$(\nabla^{(2)} w)^2 = 0. \quad (15)$$

By introducing the modulus ρ and phase ϕ of w ,

$$w = \rho e^{i\phi}, \quad (16)$$

this equation leads to the conditions

$$(\nabla^{(2)} \rho) \cdot (\nabla^{(2)} \phi) = 0, \quad (\nabla^{(2)} \rho)^2 = \rho^2 (\nabla^{(2)} \phi)^2. \quad (17)$$

It holds in fact also that

$$(\nabla^{(2)} \rho)^2 = \rho^2 (\nabla^{(2)} \phi)^2 = 1. \quad (18)$$

Conditions (17) are completely analogous to the conditions (8) in three dimensions. This leads to the natural assumption that the conditions (8) in three dimensions are just the pullbacks under the Hopf map χ of the two-dimensional conditions (17). In the sequel we want to show that this is true in a specific sense.

For this purpose, we want to re-express the above remarks in a more geometric fashion, where we introduce the metrics of the spaces under consideration and replace the gradients by exterior derivatives.

The metric on the space \mathbb{R}^2 is

$$g^{(2)} = d\rho \otimes d\rho + \rho^2 d\phi \otimes d\phi \quad (19)$$

and the dual metric is

$$G^{(2)} = \partial_\rho \otimes \partial_\rho + \frac{1}{\rho^2} \partial_\phi \otimes \partial_\phi. \quad (20)$$

The conditions (17) translate into

$$G^{(2)}(d\rho, d\phi) = 0, \quad G^{(2)}(d\rho, d\rho) = \rho^2 G^{(2)}(d\phi, d\phi) = 1 \quad (21)$$

and are obviously true.

The metric in \mathbb{R}^3 is

$$\begin{aligned} g &= q^{-2}(d\eta \otimes d\eta + d\xi \otimes d\xi + \sinh^2 \eta d\varphi \otimes d\varphi) \\ &= \frac{q^{-2}}{1+t^2}[dt \otimes dt + (1+t^2)d\xi \otimes d\xi + t^2(1+t^2)d\varphi \otimes d\varphi] \end{aligned} \quad (22)$$

where (η, ξ, φ) are the toroidal coordinates (see (4)) and the coordinate

$$t = \sinh \eta \quad (23)$$

was introduced for later convenience. The dual metric is

$$G = (1+t^2)q^2 \left(\partial_t \otimes \partial_t + \frac{1}{1+t^2} \partial_\xi \otimes \partial_\xi + \frac{1}{t^2(1+t^2)} \partial_\varphi \otimes \partial_\varphi \right). \quad (24)$$

As a next step we need the observation that a Hopf map χ introduces a fiber-bundle structure on one-point compactified \mathbb{R}^3 (or, equivalently, on S^3). Here, the fibers are the level curves of the Hopf map. The fiber has the topology of the circle S^1 , and the base space has the topology of the sphere S^2 for all Hopf maps, but the induced metric properties depend on the specific Hopf map.

Further, the Hopf map allows for a decomposition of the tangent bundle TM of the fiber bundle $M = \mathbb{R}^3$ (or S^3) into vertical and horizontal directions at each point of M . Thereby two subbundles of the full tangent bundle TM are induced, which are called the vertical distribution V and the horizontal distribution H . The vertical direction at each point points along the fiber and is spanned (in our case) by one vector field e_3 which is pushed forward to zero under the Hopf map, $\chi_* e_3 = 0$. The horizontal directions are spanned (in our case) by two vector fields e_1, e_2 , which are perpendicular to the vertical vector e_3 . Obviously, the vertical direction only depends on the Hopf map, whereas the horizontal directions depend on the bundle metric, as well. Further, we will choose all three vectors e_i to have unit length (this condition depends, of course, on the metric). This decomposition leads to an analogous decomposition at each point $p \in M$ of the cotangent space T_p^*M into a vertical direction spanned by ω_3 and horizontal directions spanned by ω_1 and ω_2 , where the ω_i are defined via

$$(\omega_i, e_j) = \delta_{ij} \quad (25)$$

and (\cdot, \cdot) denotes the canonical inner product.

Finally, the decomposition of the tangent space (and the cotangent space) into vertical and horizontal directions allows for a corresponding decomposition of the metric and its dual into a vertical and a horizontal component, $g = g_v + g_h$. They may be expressed like

$$g_h = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2, \quad g_v = \omega_3 \otimes \omega_3 \quad (26)$$

$$G_h = e_1 \otimes e_1 + e_2 \otimes e_2, \quad G_v = e_3 \otimes e_3 \quad (27)$$

in terms of the above vector fields and one-forms (observe that this notation just expresses the metric in terms of vielbeins in a coordinate-independent way).

Now we are in a position, eventually, to formulate sufficient conditions for the existence of solutions to the conditions (8).

One sufficient condition is like follows: obviously, the push-forward χ_* of the Hopf map defines an isomorphism from vectors in the horizontal distribution H of TM at points \vec{x} to vectors in TN at points $\chi(\vec{x})$ (here N is the target space manifold, i.e. \mathbb{C} or S^2 , and M is the fiber bundle). Now assume that this isomorphism is, at the same time, an isometry, i.e., the length $|\chi_* v|$ of a pushed-forward vector field $\chi_* v$ in TN w.r.t. the metric $g^{(2)}$ on N at points $\chi(\vec{x})$ is equal to the length $|v|$ of an arbitrary horizontal vector field v in H with respect to the horizontal metric g_h at points \vec{x} . Then, obviously, the lengths of one-forms remain invariant under the pull-back χ^* . For a Hopf map $w = \chi(\vec{x})$, which reads, in terms of real coordinates, like

$$\rho = S(\vec{x}), \quad \phi = \sigma(\vec{x}) \quad (28)$$

this means that the lengths should pull back like

$$|d\rho| = |dS|, \quad |d\phi| = |d\sigma| \quad (29)$$

and the target space metric $g^{(2)}$ expressed in coordinates ρ, ϕ should be identical to the horizontal metric g_h expressed in coordinates S, σ . Obviously, length relations are now conserved under the pull-back, as well,

$$|d\rho|^2 = \rho^2 |d\phi|^2 \quad \Rightarrow \quad |dS|^2 = S^2 |d\sigma|^2, \quad (30)$$

$$G^{(2)}(d\rho, d\phi) = 0 \quad \Rightarrow \quad G_h(dS, d\sigma) = 0, \quad (31)$$

which is precisely what we need in order to have solutions to the conditions (8). [Maps χ such that the push-forward $\chi_* : H \rightarrow TN$ is an isometry are called Riemannian submersions and are described at length e.g. in Ref. 11.]

It turns out that the condition on the Hopf map χ to be a Riemannian submersion is too strong for our purposes. But there is a simple generalization which does just what we want. Suppose that the lengths of horizontal vector fields are multiplied by a *common* factor at each point under the push-forward, instead of being invariant. Then the lengths of one-forms will be multiplied by a common factor under the pull-back, and this is sufficient for the conservation of the length relations (30), (31) under the pull-back. For the horizontal metric g_h and the target space metric $g^{(2)}$ this implies that they should be conformally equivalent, i.e., equal up to a local scale factor. This is precisely what happens for our solutions, as we want to demonstrate now explicitly.

First, we want to demonstrate it for the simplest Hopf map (6). We re-display the metric in \mathbb{R}^3 ,

$$g = [dt \otimes dt + (1 + t^2)d\xi \otimes d\xi + t^2(1 + t^2)d\varphi \otimes d\varphi] \quad (32)$$

where we already ignored an irrelevant local scale factor, see (22). For the Hopf map $\chi = Se^{i\sigma}$ with $S = t$, $\sigma = \xi + \varphi$, the vertical unit vector field e_3 is

$$e_3 = \frac{1}{1 + t^2}(\partial_\xi - \partial_\varphi) \quad (33)$$

(remember that $e_3(\sigma) = e_3(S) = 0$). The horizontal unit vector fields may be chosen as

$$e_1 = \partial_t, \quad e_2 = \frac{t}{1 + t^2}(\partial_\xi + t^{-2}\partial_\varphi). \quad (34)$$

The corresponding vertical and horizontal one-forms are

$$\omega_1 = dt, \quad \omega_2 = t(d\xi + d\varphi) \quad (35)$$

$$\omega_3 = d\xi - t^2 d\varphi \quad (36)$$

and the horizontal metric is

$$g_h = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 = dt \otimes dt + t^2(d\xi + d\varphi) \otimes (d\xi + d\varphi). \quad (37)$$

Obviously, this is identical to the target space metric (19) once the identification $\rho \rightarrow t$, $\phi \rightarrow \sigma = \xi + \varphi$ is made.

Now we repeat this procedure for the class of Hopf maps $S = t$, $\sigma = m\xi + n\varphi$ which are genuine Hopf maps with toroidal symmetry, but not yet the solutions (13). We find for the horizontal and vertical unit vector fields

$$e_1 = \partial_t, \quad e_2 = \frac{t}{\sqrt{(n^2 + m^2 t^2)(1 + t^2)}}(m\partial_\xi + nt^{-2}\partial_\varphi) \quad (38)$$

$$e_3 = \frac{1}{\sqrt{(n^2 + m^2 t^2)(1 + t^2)}}(n\partial_\xi - m\partial_\varphi) \quad (39)$$

and for the corresponding one-forms

$$\omega_1 = dt, \quad \omega_2 = t\sqrt{\frac{1 + t^2}{n^2 + m^2 t^2}}(md\xi + nd\varphi) \quad (40)$$

$$\omega_3 = \sqrt{\frac{1 + t^2}{n^2 + m^2 t^2}}(nd\xi - mt^2 d\varphi). \quad (41)$$

The horizontal metric now is

$$g_h = dt \otimes dt + t^2 \frac{1 + t^2}{n^2 + m^2 t^2} (md\xi + nd\varphi) \otimes (md\xi + nd\varphi) \quad (42)$$

and is not yet manifestly conformally equivalent to the target space metric. However, the horizontal metric only depends on the “horizontal” coordinates $S = t$ and $\sigma = m\xi + n\varphi$ and, therefore, certainly *is* conformally equivalent to the target space metric, because it is a well-known fact that two different metrics on a two-dimensional surface with a given topology are always conformally equivalent (see, e.g., Theorem 13.1.1 in [12]). All we have to do is to find the coordinate transformation from the horizontal coordinates (S, σ) to some new coordinates $(\tilde{S}, \tilde{\sigma})$ such that the conformal equivalence becomes manifest. This shows that the initial problem must have a solution, i.e., higher Hopf maps, related to the Hopf maps (10), which solve the static eikonal equation, *must* exist.

Explicitly, a transformation $(t, \sigma) \rightarrow (\tilde{t}(t), \sigma)$ is sufficient such that

$$g_h = \frac{t^2}{\tilde{t}^2} \frac{1 + t^2}{n^2 + m^2 t^2} [d\tilde{t} \otimes d\tilde{t} + \tilde{t}^2 (md\xi + nd\varphi) \otimes (md\xi + nd\varphi)]. \quad (43)$$

Therefore, \tilde{t} has to obey

$$(dt)^2 = \frac{(1+t^2)t^2}{(n^2+m^2t^2)\tilde{t}^2}(d\tilde{t})^2 \quad \Rightarrow \quad \frac{1}{\tilde{t}} \frac{d\tilde{t}}{dt} = \frac{1}{t} \sqrt{\frac{n^2+m^2t^2}{1+t^2}}. \quad (44)$$

Re-introducing the variable η and using $\tilde{t}(t) = \tilde{t}(\sinh \eta) \equiv f(\eta)$, the above equation (44) leads to equation (12) with the solution (13).

We want to close with two remarks. Firstly, the geometric setting developed above easily leads to more Hopf maps which solve the static eikonal equation. Obviously, the sufficient condition for the existence of a solution related to a given Hopf map is that the induced horizontal metric should be expressible - up to a local scale factor - in terms of the “horizontal” coordinates (S, σ) (where the Hopf map is $\chi = Se^{i\sigma}$). Once this condition is met, the solution can be found by transforming to new “horizontal” coordinates $(\tilde{S}, \tilde{\sigma})$ such that the horizontal metric is manifestly conformally equivalent to the target space metric. This transformation is always possible for genuine Hopf maps. The simplest example of this type for the generation of new solutions is the composition of existing solutions with maps $S^2 \rightarrow S^2$, i.e., the choice of new complex-valued functions $\chi' = F(\chi)$, where $F(\cdot)$ is a holomorphic function (e.g. a rational map), and χ is a solution. However, we already found these solutions from the symmetries of the static eikonal equation in Section 2.

Secondly, we want to remark that the above Hopf maps (10) do, in fact, provide genuine Riemannian submersions from the three-sphere S^3 to some two-dimensional target spaces with the topology of the two-sphere but, in general, metrics different from the two-sphere (except for the simplest case $m = n = 1$, which provides a Riemannian submersion from S^3 to S^2 , see, e.g., [11, 10]). This may be understood from what we said above by noting that the local scale factor q^{-2} , which is present in the metric on \mathbb{R}^3 (see (22)), and which cannot be expressed in terms of the “horizontal” coordinates alone, is absent for the metric on S^3 .

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